§4.2 Linear Transformations (Cont.)

Recall: If $T: V \to W$ is a linear transformation

$$\ker T = \{ v \text{ in } V \mid Tv = 0 \}$$

$$\text{range } T = \{ w \text{ in } W \mid w = Tv \text{ for some } v \text{ in } V \}$$

<u>Example</u> (from last time)

$T: \mathbb{P}_2 \to \mathbb{R}^2$ the linear transformation $T(p(t)) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$

Lets compute $\ker T$ and range T in a couple different ways.

<u>ker T</u>

1) division argument

$$\begin{aligned} \ker T &= \{ p(t) \mid p(0) = 0 \text{ and } p(1) = 0 \} \\ &= \{ p(t) \mid t(t-1) \text{ divides } p(t) \} \\ &= \{ p(t) = a\,t(t-1) \text{ for real \# } a \} \end{aligned}$$

deg $p(t) \leq 2$

$$\begin{aligned} &= \{ a\,t(t-1) \} \\ &= \operatorname{span}\{ t(t-1) \} \end{aligned}$$

2) coefficient argument

$\ker T = \{p(t) \text{ in } \mathbb{P}_2 \mid p(0)=0 \text{ and } p(1)=0\}$

$= \{p(t) = at^2+bt+c \mid p(0)=0 \text{ and } p(1)=0\}$

$= \{at^2+bt+c \mid c=0 \text{ and } a+b+c=0\}$

$p(0) = c$ $\qquad$ $p(1) = a+b+c$

$= \{at^2+bt+c \mid c=0 \text{ and } b=-a\}$

$= \{at^2-at \mid a \text{ real \#}\}$

$= \operatorname{span}\{t^2-t\}$

range T

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1) produce a polynomial in $\mathbb{P}_2$ that maps to all of range T (This is difficult in general!)

Let $p(t) = b+(a-b)t$ for any a, b real #'s

$\Rightarrow T(p(t)) = \begin{bmatrix} a \\ b \end{bmatrix}$ any vector in $\mathbb{R}^2$

$\Rightarrow$ range $T = \mathbb{R}^2$

This is the hard way!

2) use spanning sets

$$\text{range } T = \{ T(p(t)) \mid p(t) \text{ in } \mathbb{P}_2 \}$$

$$= \{ T(at^2+bt+c) \mid a,b,c \text{ real \#s} \}$$

$$= \left\{ \begin{bmatrix} c \\ a+b+c \end{bmatrix} \;\middle|\; a,b,c \text{ real \#s} \right\}$$

$$= \left\{ a\begin{bmatrix} 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix} + c\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

$$= \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

$$= \mathbb{R}^2 \quad \text{since } \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \text{ has a pivot in every row!}$$

This is much easier than the previous way!

In general, there are lots of ways to express kernels and ranges.

Let's use these ideas and look at other linear transformations!

Example

$T: \mathbb{P}_n \longrightarrow \mathbb{P}_{n-1}$ the differentiation transformation,

i.e. $T(p(t)) = p'(t)$. Check this is a linear transformation.

From calculus, we should find that

1) $\ker T = \{\text{constant polynomials}\} = \text{span}\{1\}$

2) $\text{range } T = \{$all polynomials of deg $\leq n-1$ that are derivatives of some polynomial$\}$

$= \mathbb{P}_{n-1}$ since polynomials have antiderivatives!

Let's verify this:

1) $\ker T = \{p(t) \mid p'(t) = 0\}$

zero polynomial

$= \{c_0 + c_1 t + \cdots + c_n t^n \mid c_1 + 2c_2 t + \cdots + c_n \cdot n t^{n-1} = 0\}$

$= \{c_0 + c_1 t + \cdots + c_n t^n \mid c_1 = c_2 = \cdots = c_n = 0\}$

$= \{c_0 \cdot 1 \mid c_0 \text{ real } \#\}$

$= \text{span}\{1\}$

2) range $T = \{ p'(t) \mid p(t) \text{ in } \mathbb{P}_n \}$

$$= \left\{ \frac{d}{dt}\left[c_0 + c_1 t + \cdots + c_n t^n\right] \mid c_0, \ldots, c_n \text{ real \#s} \right\}$$

$$= \{ c_1 + 2c_2 t + 3c_3 t^2 + \cdots + n c_n t^{n-1} \mid c_1, \ldots, c_n \text{ real \#s} \}$$

$$= \text{span}\{1, t, t^2, \ldots, t^{n-1}\}$$

$$= \mathbb{P}_{n-1}$$

$\{1, t, t^2, \ldots, t^n\}$ is a basis for $\mathbb{P}_n$!

Example (exercise)

Let $T: \mathbb{P}_n \longrightarrow \mathbb{P}_{n+1}$ be the integration transformation,
i.e. $T(p(t)) = \int p(t)\, dt$ without a "$+ C$".

1) check this is a linear transformation

2) what is $\ker T$?

3) what is range T ?

Example

Let $T: M_{2\times 2} \rightarrow M_{2\times 2}$ be the transformation given by $T(A) = A^T + A$

what is $\ker T$?

$$\ker T = \{ 2\times 2 \; A \mid A = -A^T \}$$

$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -a & -c \\ -b & -d \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \begin{matrix} a = -a & b = -c \\ d = -d & c = -b \end{matrix} \right\}$$

$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \begin{matrix} a = 0 \\ d = 0 \end{matrix} \quad b = -c \right\}$$

$$= \left\{ \begin{pmatrix} 0 & -c \\ c & 0 \end{pmatrix} \;\middle|\; c \text{ real } \# \right\}$$

$$= \text{span} \left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$$

$$\text{range } T = \{ A + A^T \mid A \ 2\times 2 \}$$

$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a & c \\ b & d \end{pmatrix} \;\middle|\; a,b,c,d \text{ real \#s} \right\}$$

$$= \left\{ \begin{pmatrix} 2a & b+c \\ b+c & 2d \end{pmatrix} \;\middle|\; a,b,c,d \right\}$$

$$= \left\{ 2a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$= \text{span}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

<u>Exercise</u>: Do the same thing with

$T: M_{2\times 2} \to M_{2\times 2}$ where $T(A) = A - A^T$

are the results similar?